Euler Circuits in Graphs

A dinner party problem

In a dinner party, if count (mutual) acquaintances or (mutual) strangers, one may find the following interesting facts:

A dinner party problem

In a dinner party, if count (mutual) acquaintances or (mutual) strangers, one may find the following interesting facts:

1. everybody looks at the number of his/her acquaintances and if add all these numbers up, it is an even number

A dinner party problem

In a dinner party, if count (mutual) acquaintances or (mutual) strangers, one may find the following interesting facts:

1. everybody looks at the number of his/her acquaintances and if add all these numbers up, it is an even number
2. the number of people with an odd number of acquaintances is even

A dinner party problem

In a dinner party, if count (mutual) acquaintances or (mutual) strangers, one may find the following interesting facts:

1. everybody looks at the number of his/her acquaintances and if add all these numbers up, it is an even number
2. the number of people with an odd number of acquaintances is even
3. if there are at least 6 people, then there are a subset of 3 people who are all either mutual acquaintances, or mutual strangers

A dinner party problem

In a dinner party, if count (mutual) acquaintances or (mutual) strangers, one may find the following interesting facts:

1. everybody looks at the number of his/her acquaintances and if add all these numbers up, it is an even number
2. the number of people with an odd number of acquaintances is even
3. if there are at least 6 people, then there are a subset of 3 people who are all either mutual acquaintances, or mutual strangers

Here is why [3]:

A dinner party problem

In a dinner party, if count (mutual) acquaintances or (mutual) strangers, one may find the following interesting facts:

1. everybody looks at the number of his/her acquaintances and if add all these numbers up, it is an even number
2. the number of people with an odd number of acquaintances is even
3. if there are at least 6 people, then there are a subset of 3 people who are all either mutual acquaintances, or mutual strangers

Here is why [3]:

- Denote each person as a vertex (node), and draw a line between two people if they are mutual acquaintances.

A dinner party problem

In a dinner party, if count (mutual) acquaintances or (mutual) strangers, one may find the following interesting facts:

1. everybody looks at the number of his/her acquaintances and if add all these numbers up, it is an even number
2. the number of people with an odd number of acquaintances is even
3. if there are at least 6 people, then there are a subset of 3 people who are all either mutual acquaintances, or mutual strangers

Here is why [3]:

- Denote each person as a vertex (node), and draw a line between two people if they are mutual acquaintances.
- Start from any vertex (labeled as u), at least three vertices have edges (or no edges) with u, then we have a triangle (or empty triangle) if there are any edge between the three vertices, and they form an empty triangle (or triangle) if they have no edge at all.

Seven Bridge Problem

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges.

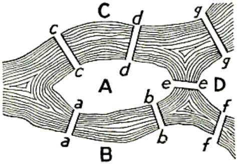


FIGURE 98. *Geographic Map: The Königsberg Bridges.*

The problem was to find a walk through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time.

Seven Bridge Problem

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges.

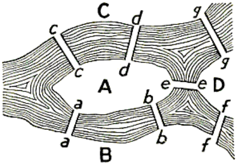
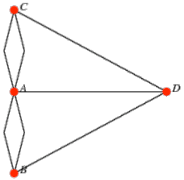


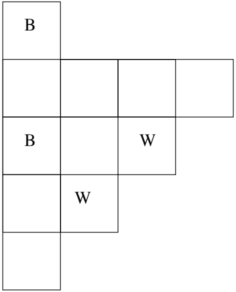
FIGURE 98. *Geographic Map: The Königsberg Bridges.*



The problem was to find a walk through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time.

Knights on the chessboard

There are two black knights and two white knights on the partial chess board. Switch the black knights with the white knights (no two knights at same position any time). What is the minimum number of steps to do this?



Basics on graphs

- A graph G consists of vertices (nodes) and edges (lines) between vertices.

Basics on graphs

- A graph G consists of vertices (nodes) and edges (lines) between vertices.

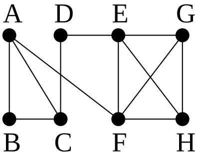
- In practice, vertices are objects (people, cities, et al), and edges are relations between objects (friendship, flights, et al).

Basics on graphs

- A graph G consists of vertices (nodes) and edges (lines) between vertices.
- In practice, vertices are objects (people, cities, et al), and edges are relations between objects (friendship, flights, et al).
- In a graph, every vertex x has neighbors—those vertices having an edge with x. We use $N(x)$ to denote the neighbors of x. The degree of x is the number of edges incident to x.

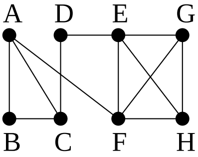
An example

Ex: In the graph below, the vertices represent houses and two vertices joined by an edge if it is possible to drive between the two houses in under 10 minutes.



An example

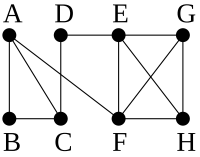
Ex: In the graph below, the vertices represent houses and two vertices joined by an edge if it is possible to drive between the two houses in under 10 minutes.



- How many vertices and how many edges do the graph have?

An example

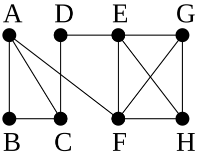
Ex: In the graph below, the vertices represent houses and two vertices joined by an edge if it is possible to drive between the two houses in under 10 minutes.



- How many vertices and how many edges do the graph have?
- What are the degrees of the vertices in this graph?

An example

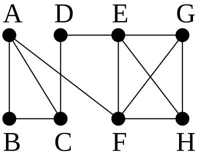
Ex: In the graph below, the vertices represent houses and two vertices joined by an edge if it is possible to drive between the two houses in under 10 minutes.



- How many vertices and how many edges do the graph have?
- What are the degrees of the vertices in this graph?
- What is the meaning that A has degree three?

An example

Ex: In the graph below, the vertices represent houses and two vertices joined by an edge if it is possible to drive between the two houses in under 10 minutes.



- How many vertices and how many edges do the graph have?
- What are the degrees of the vertices in this graph?
- What is the meaning that A has degree three?
- For which houses, if any, is it possible to drive to all the other houses in less than 20 minutes?

Basics on graph—-Continue

- A walk in a graph is a sequence of vertices and edges in a graph such that each vertex belongs to the edge before and after it.

Basics on graph—-Continue

- A walk in a graph is a sequence of vertices and edges in a graph such that each vertex belongs to the edge before and after it.

Ex: $A, AC, C, CD, D, DE, E, EH, H, HF, F, FE, E, EG, G$ is a walk from A to G

Basics on graph—-Continue

- A walk in a graph is a sequence of vertices and edges in a graph such that each vertex belongs to the edge before and after it.

Ex: $A, AC, C, CD, D, DE, E, EH, H, HF, F, FE, E, EG, G$ is a walk from A to G

- A path is a walk without repeated vertices

Basics on graph—-Continue

- A walk in a graph is a sequence of vertices and edges in a graph such that each vertex belongs to the edge before and after it.

Ex: $A, AC, C, CD, D, DE, E, EH, H, HF, F, FE, E, EG, G$ is a walk from A to G

- A path is a walk without repeated vertices
- A circuit is a closed walk, i.e., a walk starting and ending at the same vertex

Basics on graph—-Continue

- A walk in a graph is a sequence of vertices and edges in a graph such that each vertex belongs to the edge before and after it.

Ex: $A, AC, C, CD, D, DE, E, EH, H, HF, F, FE, E, EG, G$ is a walk from A to G

- A path is a walk without repeated vertices
- A circuit is a closed walk, i.e., a walk starting and ending at the same vertex

Ex: Find a path from A to G of length 2. What is the meaning of the path?

Basics on graph—-Continue

- A walk in a graph is a sequence of vertices and edges in a graph such that each vertex belongs to the edge before and after it.

Ex: $A, AC, C, CD, D, DE, E, EH, H, HF, F, FE, E, EG, G$ is a walk from A to G

- A path is a walk without repeated vertices
- A circuit is a closed walk, i.e., a walk starting and ending at the same vertex

Ex: Find a path from A to G of length 2. What is the meaning of the path?

- A graph is connected if there is a path between any two vertices

Basics on graph—-Continue

- A walk in a graph is a sequence of vertices and edges in a graph such that each vertex belongs to the edge before and after it.

Ex: $A, AC, C, CD, D, DE, E, EH, H, HF, F, FE, E, EG, G$ is a walk from A to G

- A path is a walk without repeated vertices
- A circuit is a closed walk, i.e., a walk starting and ending at the same vertex

Ex: Find a path from A to G of length 2. What is the meaning of the path?

- A graph is connected if there is a path between any two vertices

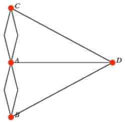
Ex: Can you draw a non-connected graph? What's the meaning of that?

Seven bridge problem revisit—-Euler circuit

- What do we need in the graph?

Seven bridge problem revisit—-Euler circuit

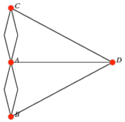
- What do we need in the graph?



- We need a circuit which covers all the edges once and exactly once. Such a circuit is called an Euler Circuit.

Seven bridge problem revisit—-Euler circuit

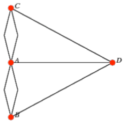
- What do we need in the graph?



- We need a circuit which covers all the edges once and exactly once. Such a circuit is called an Euler Circuit.
- If a graph has an Euler circuit,

Seven bridge problem revisit—-Euler circuit

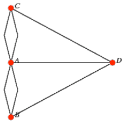
- What do we need in the graph?



- We need a circuit which covers all the edges once and exactly once. Such a circuit is called an Euler Circuit.
- If a graph has an Euler circuit,
  - the graph has to be connected.

Seven bridge problem revisit—-Euler circuit

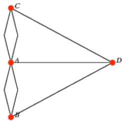
- What do we need in the graph?



- We need a circuit which covers all the edges once and exactly once. Such a circuit is called an Euler Circuit.
- If a graph has an Euler circuit,
  - the graph has to be connected.
  - every time the circuit visits a vertex, it will take two edges incident to the vertex. So in order for the circuit cover all the edges, every vertex has to have even edges incident to it. That is, all the vertices have even degrees.

Seven bridge problem revisit—-Euler circuit

- What do we need in the graph?



- We need a circuit which covers all the edges once and exactly once. Such a circuit is called an Euler Circuit.
- If a graph has an Euler circuit,
  - the graph has to be connected.
  - every time the circuit visits a vertex, it will take two edges incident to the vertex. So in order for the circuit cover all the edges, every vertex has to have even edges incident to it. That is, all the vertices have even degrees.
- So if a graph is not connected, OR some vertex has an odd degree, then it has NO Euler circuit.

Euler's Theorem

- Euler's Theorem: A connected graph with only even degrees has an Euler circuit.

Euler's Theorem

- Euler's Theorem: A connected graph with only even degrees has an Euler circuit.

- Let's see how to find an Euler circuit in a connected even graph:

Euler's Theorem

- Euler's Theorem: A connected graph with only even degrees has an Euler circuit.

- Let's see how to find an Euler circuit in a connected even graph:
  - Start from any vertex, find a circuit C without repeated edges

Euler's Theorem

- Euler's Theorem: A connected graph with only even degrees has an Euler circuit.

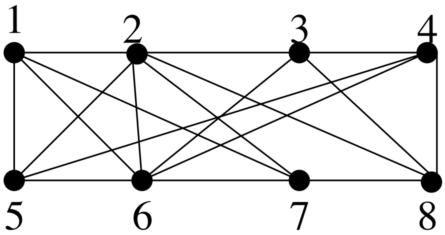
- Let's see how to find an Euler circuit in a connected even graph:
  - Start from any vertex, find a circuit C without repeated edges
  - Remove the edges in C from the graph, and find Euler circuits from each component of the remaining graph

Euler's Theorem

- Euler's Theorem: A connected graph with only even degrees has an Euler circuit.
- Let's see how to find an Euler circuit in a connected even graph:
  - Start from any vertex, find a circuit C without repeated edges
  - Remove the edges in C from the graph, and find Euler circuits from each component of the remaining graph
  - Combine C with the Euler circuits to get an Euler circuit in the original graph

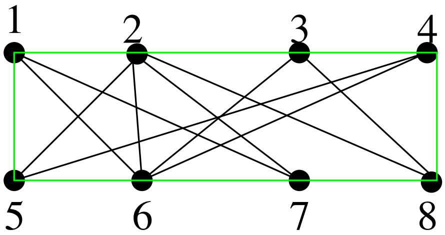
Example: Euler circuit

- Find an Euler circuit in the following (left) graph



Example: Euler circuit

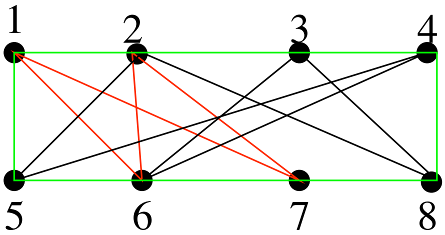
- Find an Euler circuit in the following (left) graph



- 12, 23, 34, 48, 87, 76, 65, 51

Example: Euler circuit

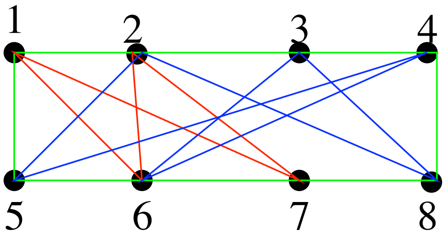
- Find an Euler circuit in the following (left) graph



- 12, 23, 34, 48, 87, 76, 65, 51
- 16, 62, 27, 71

Example: Euler circuit

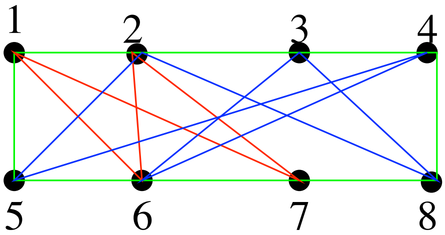
- Find an Euler circuit in the following (left) graph



- 12, 23, 34, 48, 87, 76, 65, 51
- 16, 62, 27, 71
- 52, 28, 83, 36, 64, 45

Example: Euler circuit

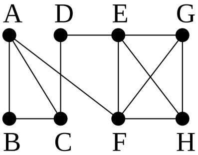
- Find an Euler circuit in the following (left) graph



- 12, 23, 34, 48, 87, 76, 65, 51
- 16, 62, 27, 71
- 52, 28, 83, 36, 64, 45
- Euler circuit:
  16, 62, 27, 71, 12, 23, 34, 48, 87, 76, 65, 52, 28, 83, 36, 64, 45, 51

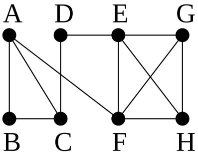
Optimization problems in non-Euler graphs

Ex: Does the following have an Euler circuit?



Optimization problems in non-Euler graphs

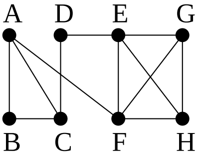
Ex: Does the following have an Euler circuit?



- How many edges do we need to repeat to cover all the edges?

Optimization problems in non-Euler graphs

Ex: Does the following have an Euler circuit?

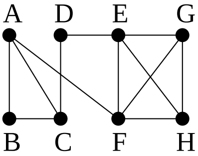


- How many edges do we need to repeat to cover all the edges?

  —— we have to repeat at least two edges (for example, repeat AC and GH)

Optimization problems in non-Euler graphs

Ex: Does the following have an Euler circuit?

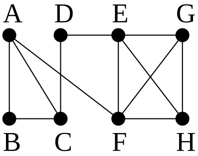


- How many edges do we need to repeat to cover all the edges?

  —— we have to repeat at least two edges (for example, repeat AC and GH)
- Observe that the vertices without even degrees are A, C, G, and H. Is this a coincidence?

Optimization problems in non-Euler graphs

Ex: Does the following have an Euler circuit?



- How many edges do we need to repeat to cover all the edges?

  —— we have to repeat at least two edges (for example, repeat AC and GH)
- Observe that the vertices without even degrees are A, C, G, and H. Is this a coincidence?
- Not really! If add edges AC and GH to the original graph, we will get a graph with only even degrees, thus get an Euler circuit.

Facts on graphs

- In order to do this, we should be able to pair up the vertices with odd degrees.

Facts on graphs

- In order to do this, we should be able to pair up the vertices with odd degrees.
- In other words, there are even number of odd-degree vertices in a graph. Is this always true?

Facts on graphs

- In order to do this, we should be able to pair up the vertices with odd degrees.
- In other words, there are even number of odd-degree vertices in a graph. Is this always true?
- Yes! Here are two facts about graphs and the first one implies the second one

Facts on graphs

- In order to do this, we should be able to pair up the vertices with odd degrees.
- In other words, there are even number of odd-degree vertices in a graph. Is this always true?
- Yes! Here are two facts about graphs and the first one implies the second one
  - The sum of all degrees is even

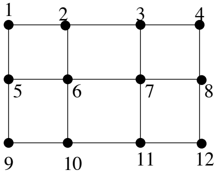
Facts on graphs

- In order to do this, we should be able to pair up the vertices with odd degrees.
- In other words, there are even number of odd-degree vertices in a graph. Is this always true?
- Yes! Here are two facts about graphs and the first one implies the second one
  - The sum of all degrees is even
  - The number of odd-degree vertices is even

Facts on graphs

- In order to do this, we should be able to pair up the vertices with odd degrees.
- In other words, there are even number of odd-degree vertices in a graph. Is this always true?
- Yes! Here are two facts about graphs and the first one implies the second one
  - The sum of all degrees is even
  - The number of odd-degree vertices is even
- Now we know the number of odd-degree vertices is always even, so in order to create an Euler circuit in a non-Eulerian graph, we just need to identify the odd-degree vertices, pair them up, add or repeat the corresponding edges.

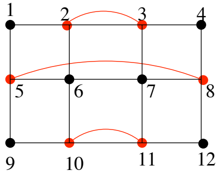
Ex. Find an Euler circuit in the following (left) graph with fewest repeated edges



Ex. Find an Euler circuit in the following (left) graph with fewest repeated edges

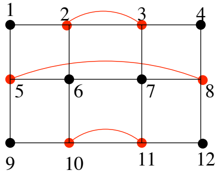
- Identify the odd-degree vertices: 2, 3, 5, 8, 10, 12

Ex. Find an Euler circuit in the following (left) graph with fewest repeated edges



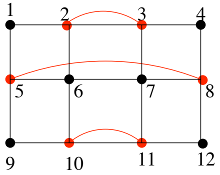
- Identify the odd-degree vertices: 2, 3, 5, 8, 10, 12
- Pair up the vertices so that the total distance between pairs is smallest: (2, 3), (5, 8), and (10, 11) and add the edges

Ex. Find an Euler circuit in the following (left) graph with fewest repeated edges



- Identify the odd-degree vertices: 2, 3, 5, 8, 10, 12
- Pair up the vertices so that the total distance between pairs is smallest: (2, 3), (5, 8), and (10, 11) and add the edges
- In the new graph, find an Euler circuit:
  12, 23, 34, 48, 87, 76, 65, 59, 9(10), (10)6, 62, 23, 37, 7(11), (11)(10), (10)(11), (11)(12), (12)8, 85, 51

Ex. Find an Euler circuit in the following (left) graph with fewest repeated edges



- Identify the odd-degree vertices: 2, 3, 5, 8, 10, 12
- Pair up the vertices so that the total distance between pairs is smallest: (2, 3), (5, 8), and (10, 11) and add the edges
- In the new graph, find an Euler circuit:
  12, 23, 34, 48, 87, 76, 65, 59, 9(10), (10)6, 62, 23, 37, 7(11), (11)(10), (10)(11), (11)(12), (12)8, 85, 51
- Replace the added edges (red edges) with paths in the original graph:
  12, 23, 34, 48, 87, 76, 65, 59, 9(10), (10)6, 62, 23, 37, 7(11), (11)(10), (10)(11), (11)(12), (12)8, 87, 76, 65, 51

General Chinese Postman Problem

- Chinese Postman Problem: find the a circuit in a graph to cover all the edges with fewest repeated edges.

General Chinese Postman Problem

- Chinese Postman Problem: find the a circuit in a graph to cover all the edges with fewest repeated edges.
- We have already known how to solve it:

General Chinese Postman Problem

- Chinese Postman Problem: find the a circuit in a graph to cover all the edges with fewest repeated edges.
- We have already known how to solve it:
  - identify odd-degree vertices

General Chinese Postman Problem

- Chinese Postman Problem: find the a circuit in a graph to cover all the edges with fewest repeated edges.
- We have already known how to solve it:
  - identify odd-degree vertices
  - pair them up and add new edges between them

General Chinese Postman Problem

- Chinese Postman Problem: find the a circuit in a graph to cover all the edges with fewest repeated edges.
- We have already known how to solve it:
  - identify odd-degree vertices
  - pair them up and add new edges between them
  - replace the new edges by paths between the endpoints of the edges

General Chinese Postman Problem

- Chinese Postman Problem: find the a circuit in a graph to cover all the edges with fewest repeated edges.
- We have already known how to solve it:
  - identify odd-degree vertices
  - pair them up and add new edges between them
  - replace the new edges by paths between the endpoints of the edges
- There are further questions to ask:

General Chinese Postman Problem

- Chinese Postman Problem: find the a circuit in a graph to cover all the edges with fewest repeated edges.
- We have already known how to solve it:
  - identify odd-degree vertices
  - pair them up and add new edges between them
  - replace the new edges by paths between the endpoints of the edges
- There are further questions to ask:
  - There are a lot of ways to pair the vertices up. How to pair them up?

General Chinese Postman Problem

- Chinese Postman Problem: find the a circuit in a graph to cover all the edges with fewest repeated edges.
- We have already known how to solve it:
  - identify odd-degree vertices
  - pair them up and add new edges between them
  - replace the new edges by paths between the endpoints of the edges
- There are further questions to ask:
  - There are a lot of ways to pair the vertices up. How to pair them up?
  - There are a lot of different paths between two vertices. Which one to choose?

General Chinese Postman Problem

- Chinese Postman Problem: find the a circuit in a graph to cover all the edges with fewest repeated edges.
- We have already known how to solve it:
  - identify odd-degree vertices
  - pair them up and add new edges between them
  - replace the new edges by paths between the endpoints of the edges
- There are further questions to ask:
  - There are a lot of ways to pair the vertices up. How to pair them up?
  - There are a lot of different paths between two vertices. Which one to choose?
- We will need shortest path algorithm and minimum-weighted matching algorithm to answer these two questions.